

Lorentz-violating effects in the spin-1/2 Aharonov-Casher problem

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The effects of a Lorentz symmetry violating background vector on the Aharonov-Casher bound and scattering scenarios is considered. Using an approach based on the self-adjoint extension method, an expression for the bound state energies is obtained in terms of the physics of the problem by determining the self-adjoint extension parameter. We found that there is an additional scattering for any value of the self-adjoint extension parameter and bound states for negative values of this parameter. By comparing the bound state and scattering results the self-adjoint extension parameter is determined. Expressions for the bound state energies, phase-shift and the scattering matrix are explicitly determined in terms of the self-adjoint extension parameter. The expression obtained for the scattering amplitude reveals that the helicity is not conserved in this scenario.

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I. INTRODUCTION

The standard model extension (SME) [1–3] has been an usual framework for investigating signals of Lorentz violation in physical systems and has inspired a great deal of investigations in this theme in recent years. The interest in this issue appears in the different contexts, such as field theory [4–20], gravitation [21, 22], aspects on the gauge sector of the SME [23–26], quantum electrodynamics [27–31], nonrelativistic quantum dynamics and topological phase [32–38], and astrophysics [39–41]. These many contributions have elucidated effects induced by Lorentz violation and the SME has also been used as a framework to propose Lorentz violating in CPT probing experiments, which have amount to the imposition of stringent upper bounds on the Lorentz-violating (LV) coefficients [42–44].

The physical properties of the physical systems can be accessed by including in all sectors of the minimal standard model LV terms. The LV terms are generated as vacuum expectation values of tensors defined in a high energy scale. By carefully analyzing the sectors of the SME some authors have specialized in introducing news nonminimal couplings between fermionic and gauge fields in the context of the Dirac equation [45, 46]. In the fermion sector, for example, this violation is implemented by introducing two CPT-odd terms, $V_\mu \bar{\psi} \gamma^\mu \psi$, $W_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi$, where V_μ , W_μ are the LV backgrounds.

In this paper, we reexamine the Aharonov-Casher (AC) problem in a CPT-odd Lorentz-violating background addressed in Ref. [47]. We analyze the calculations of the derivation of the nonrelativistic Hamiltonian,

obtaining the correct Hamiltonian and supplying a condition that establishes the exact equivalence between the original spin-1/2 AC and LV-AC effects. We also solve the scattering and bound state problems of the model using the physical regularization scheme [48, 49] based on the self-adjoint extension method proposed in Refs. [50, 51].

The work is outlined in the following way: In Section II, we derive the equation of planar motion in order to study the physical implications of the LV background on the spin-1/2 AC problem. We also obtain a condition that establishes the exact equivalence between the original spin-1/2-AC and LV-AC effects. The Section III is devoted to the study of the LV Hamiltonian via the self-adjoint extension technique and are presented some important properties of the LV wave function. In Section IV the bound state energy is determined in terms of the physics of the problem. In Section V are addressed the scattering and bound state problems within the framework of the LV Schrödinger-Pauli equation. Expressions for the bound state energies, phase-shift and scattering matrix are computed and all them are explicitly described in terms of the physical condition of the problem. The self-adjoint extension parameter is also derived in terms of the physical parameters. At the end, we make a detailed analysis of the helicity conservation's problem in the present framework. In Section VI we give our conclusions and remarks.

II. THE EQUATION OF MOTION

In this section, we derive the equation of motion that governs the dynamics of a spin-1/2 neutral particle in a radial electric field and a LV background vector. We begin with the (3+1)-dimensional Dirac equation with a

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LV and CPT-odd nonminimal coupling between fermions and the gauge field as proposed in Ref. [45] in the form ($\hbar = c = 1$ and signature $(+ - - -)$)

$$[i\gamma^\mu D_\mu - M]\Psi = 0, \quad (1)$$

with

$$D_\mu = \partial_\mu + ieA_\mu + igV^\nu \tilde{F}_{\mu\nu}, \quad (2)$$

where Ψ is the fermion spinor of four-component, $V^\mu = (V_0, \mathbf{V})$ is the Carroll-Field-Jackiw four-vector, e is the electric charge, g is a constant that measures the non-minimal coupling magnitude. The electromagnetic field tensor is given by

$$\begin{aligned} F^{\mu\nu} &= -F^{\nu\mu}, \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\ &= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \end{aligned} \quad (3)$$

where $A^\mu = (A^0, \mathbf{A})$ is the 4-vector potential. By using the Levi-Civita's antisymmetric symbol $\varepsilon^{\mu\nu\rho\sigma}$ (with $\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$) to be equal to 1 or -1 according to whether $(\mu\nu\rho\sigma)$ is an even or odd permutation of $(0, 1, 2, 3)$ and zero otherwise, we can obtain the dual tensor

$$\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \quad (4)$$

The tensor $\tilde{F}_{\mu\nu}$ is obtained directly from $\tilde{F}^{\mu\nu}$ as

$$\begin{aligned} \tilde{F}_{\mu\nu} &= g_{\mu\gamma}\tilde{F}^{\gamma\delta}g_{\delta\nu}, \\ &= \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & E^3 & -E^2 \\ -B^2 & -E^3 & 0 & E^1 \\ -B^3 & E^2 & -E^1 & 0 \end{pmatrix}. \end{aligned} \quad (5)$$

From the above results, we identify

$$\tilde{F}_{0i} = B^i = -B_i, \quad (6)$$

$$\tilde{F}_{ij} = \varepsilon^{ijk}E^k = \varepsilon_{ijk}E_k, \quad (7)$$

so that the Dirac equation (1) can be written more explicitly as

$$\begin{aligned} \bar{\mathcal{E}}\psi &= \boldsymbol{\alpha} \cdot [\mathbf{p} - e\mathbf{A} - gV^0\mathbf{B} - g(\mathbf{V} \times \mathbf{E})]\psi \\ &+ (eA_0 + g\mathbf{V} \cdot \mathbf{B} + \beta M)\psi, \end{aligned} \quad (8)$$

where

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad (9)$$

are the standard Dirac matrices and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Since we are only interested in planar dynamics of the AC problem, then we specialize to the case $p_3 = 0$ in Eq. (8) and consider only the components \tilde{F}_{ij} of $\tilde{F}_{\mu\nu}$. In this case, the relevant equation is the planar Dirac equation

$$\{\boldsymbol{\alpha} \cdot [\mathbf{p} - g(\mathbf{V} \times \mathbf{E})] + \beta M\}\psi = \bar{\mathcal{E}}\psi. \quad (10)$$

Equation (10) in the nonrelativistic limit is found to be

$$\hat{H}\psi = \mathcal{E}\psi, \quad (11)$$

where

$$\hat{H} = \frac{1}{2M}[\mathbf{p} - g(\mathbf{V} \times \mathbf{E})]^2 - \frac{1}{2M}gs\boldsymbol{\sigma} \cdot [\nabla \times (\mathbf{V} \times \mathbf{E})], \quad (12)$$

is the Hamiltonian operator. The field configuration (in cylindrical coordinates) is given by

$$\mathbf{E} = \frac{2\lambda}{r}\hat{\mathbf{r}}, \quad \nabla \cdot \mathbf{E} = 2\lambda\frac{\delta(r)}{r}, \quad V^\mu = (0, 0, 0, V_z), \quad (13)$$

where \mathbf{E} , is the electric field generated by an infinite charged filament and λ is the charge density along the z -axis.

After substitution of Eq. (13) into (12), we find

$$\hat{H} = \frac{1}{2M} \left[\hat{H}_0 - s\eta\sigma_z\frac{\delta(r)}{r} \right], \quad (14)$$

with

$$\hat{H}_0 = \left(\frac{1}{i}\nabla - \eta\frac{\hat{\varphi}}{r} \right)^2, \quad (15)$$

and

$$\eta = 2\lambda gV_z, \quad (16)$$

is the coupling constant of the $\delta(r)/r$ potential.

The Hamiltonian in Eq. (14) governs the quantum dynamics of a spin-1/2 neutral particle with a radial electric field, i.e., a spin-1/2 AC problem, with $g\mathbf{V}$ playing the role of a nontrivial magnetic dipole moment. Also, note the presence of a δ function which is singular at the origin in Eq. (14). This makes the problem more complicated to be solved. Such kind of point interaction potential can then be addressed by the self-adjoint extension approach [50–52], which will be used for studying the scattering and bound state scenarios.

Now, it is instructive to take look at the exact equivalence between the LV-AC problem and the usual AC problem. The Hamiltonian of the usual AC problem for a neutral particle with magnetic moment $\boldsymbol{\mu}$ can be written as [53]

$$\hat{H}_{AC} = [\mathbf{p} + s(\boldsymbol{\mu} \times \mathbf{E})]^2 + \frac{1}{2M}\mu\sigma_3(\nabla \cdot \mathbf{E}), \quad (17)$$

which upon substitution of (13), it assumes the form

$$\hat{H}_{AC} = \frac{1}{2M} \left(\frac{1}{i} \nabla + s\eta_{AC} \frac{\hat{\varphi}}{r} \right)^2 + \frac{1}{2M} \eta_{AC} \sigma_z \frac{\delta(r)}{r}, \quad (18)$$

where

$$\eta_{AC} = 2\lambda\mu. \quad (19)$$

It is worth of mention that the operator in (18) is formally the same as the two-dimensional spin-1/2 Aharonov-Bohm Hamiltonian, with the delta function playing the role of the Zeeman interaction between the spin and the magnetic flux tube [54]. This later problem has been solved in full details in Refs. [54–57] (see also [52] and references therein.)

We can observe in Eq. (17) that the magnetic moment of the particle, $\boldsymbol{\mu} = \mu\boldsymbol{\sigma}$, it is an spinor quantity. On the other hand, in the LV-AC Hamiltonian (Eq. (12)), the quantity $g\mathbf{V}$ has vectorial character. In this way, the equivalence between the effects is achieved by replacing

$$gV_z \rightarrow -\mu s, \quad (20)$$

directly in Eq. (14). Thus, we have established an exact equivalence between the usual AC and LV-AC effects.

III. SELF-ADJOINT EXTENSION ANALYSIS

An operator \mathcal{O} , with domain $\mathcal{D}(\mathcal{O})$, is said to be self-adjoint if and only if $\mathcal{D}(\mathcal{O}^\dagger) = \mathcal{D}(\mathcal{O})$ and $\mathcal{O}^\dagger = \mathcal{O}$. In order to determine all self-adjoint extensions of (15), making use of the underlying rotational symmetry expressed by the fact that $[\hat{H}, \hat{J}_z] = 0$, where $\hat{J}_z = -i\partial/\partial\varphi + \sigma_z/2$ is the total angular momentum operator in the z -direction. Furthermore, the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2)$ is decomposed with respect to the total angular momentum $\mathfrak{H} = \mathfrak{H}_r \otimes \mathfrak{H}_\varphi$, where $\mathfrak{H}_r = L^2(\mathbb{R}^+, r dr)$ and $\mathfrak{H}_\varphi = L^2(\mathcal{S}^1, d\varphi)$, with \mathcal{S}^1 denoting the unit sphere in \mathbb{R}^2 . So, it is possible to express the eigenfunctions of the two dimensional Hamiltonian in terms of the eigenfunctions of \hat{J}_z

$$\Psi(r, \varphi) = \begin{pmatrix} \psi_m(r) e^{i(m_j - 1/2)\varphi} \\ \chi_m(r) e^{i(m_j + 1/2)\varphi} \end{pmatrix}, \quad (21)$$

with $m_j = m + 1/2 = \pm 1/2, \pm 3/2, \dots$, and $m \in \mathbb{Z}$. By inserting Eq. (21) into Eq. (11) the Schrödinger-Pauli equation for $\psi_m(r)$ is found to be ($k^2 = 2ME$)

$$H\psi_m(r) = k^2\psi_m(r), \quad (22)$$

where

$$H = H_0 - s\eta \frac{\delta(r)}{r}, \quad (23)$$

and

$$H_0 = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(m - \eta)^2}{r^2}. \quad (24)$$

The self-adjoint extension approach consists, essentially, in extending the domain $\mathcal{D}(H_0)$ to match $\mathcal{D}(H_0^\dagger)$ and therefore turning H_0 into a self-adjoint operator. To do so, we must find the deficiency subspaces, N_\pm , with dimensions n_\pm , which are called deficiency indices of H_0 [58]. A necessary and sufficient condition for H_0 being essentially self-adjoint is that $n_+ = n_- = 0$. On the other hand, if $n_+ = n_- \geq 1$, then H_0 will have an infinite number of self-adjoint extensions parametrized by unitary operators $U_\theta : N_+ \rightarrow N_-$, with $\theta \in [0, 2\pi)$. In order to find the deficiency subspaces of H_0 in \mathfrak{H}_r , we must solve the eigenvalue equation

$$H_0^\dagger \psi_\pm = \pm i k_0^2 \psi_\pm, \quad (25)$$

where $k_0^2 \in \mathbb{R}$ was introduced for dimensional reasons. Since $H_0^\dagger = H_0$, the solutions of Eq. (25) which vanishes at the infinite are the modified Bessel functions of second kind (up to a constant)

$$\psi_\pm = K_{|m-\eta|}(\varepsilon_\pm r), \quad (26)$$

with $\varepsilon_\pm = e^{\mp i\pi/4} k_0$. The solutions ψ_\pm are normalizable if and only if $|m - \eta| < 1$. The dimension of such deficiency subspace is thus $(n_+, n_-) = (1, 1)$. According to the von Neumann-Krein theory, all self-adjoint extensions $H_{\theta,0}$ of H_0 are given by the one-parameter family

$$\mathcal{D}(H_{\theta,0}) = \mathcal{D}(H_0) \oplus (I + U_\theta)N_+. \quad (27)$$

Thus, $\mathcal{D}(H_{\theta,0})$ in \mathfrak{H}_r is given by the set of functions [58]

$$\psi_\theta(r) = \psi_m(r) + c [K_{|m-\eta|}(\varepsilon_+ r) + e^{i\theta} K_{|m-\eta|}(\varepsilon_- r)], \quad (28)$$

where $\psi_m(r)$, with $\psi_m(0) = \dot{\psi}_m(0) = 0$ ($\dot{\psi} \equiv d\psi/dr$), is a regular wave function, $c \in \mathbb{C}$ and the number $\theta \in [0, 2\pi)$ represents a choice for the boundary condition. Using the unitary operator $U : L^2(\mathbb{R}^+, r dr) \rightarrow L^2(\mathbb{R}^+, dr)$, given by $(U\xi)(r) = r^{1/2}\xi(r)$, the operator H_0 reads

$$\tilde{H}_0 = U H_0 U^{-1} = -\frac{d^2}{dr^2} - \frac{(m - \eta)^2 - 1/4}{r^2}. \quad (29)$$

By standard results, the radial operator \tilde{H}_0 is essentially self-adjoint for $|m - \eta| \geq 1$, while for $|m - \eta| < 1$ it admits an one-parameter family of self-adjoint extensions [58]. This statement can be understood based in Eq. (26), because for $|m - \eta| \geq 1$ the right hand side is not in \mathfrak{H}_r at 0, while it is in \mathfrak{H}_r for $|m - \eta| < 1$. To characterize the one-parameter family of the self-adjoint extension, we will use the KS [51] and BG [50] approaches, both base in boundary condition at the origin, as we explain below.

A. KS self-adjoint extension approach

Following the Ref. [51], in the KS approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for the problem H_0 plus

self-adjoint extension, i.e., one considers the zero-energy solutions ψ_0 and $\psi_{\theta,0}$ for H and H_0 , respectively,

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\eta)^2}{r^2} + s\eta \frac{\delta(r)}{r} \right] \psi_0 = 0, \quad (30)$$

and

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\eta)^2}{r^2} \right] \psi_{\theta,0} = 0. \quad (31)$$

A fitting value for the number θ is determined by the boundary condition at the origin

$$a \frac{\dot{\psi}_0}{\psi_0} \Big|_{r=a} = a \frac{\dot{\psi}_{\theta,0}}{\psi_{\theta,0}} \Big|_{r=a}. \quad (32)$$

where a is a very small radius, being smaller than the Compton wave length λ_C of the electron [59], which comes from the regularization of the δ function [48, 49]. The present approach has the advantage of yielding the number θ in terms of the physics of the problem, but is only applicable for determination of bound states, being not appropriate for dealing with scattering problems.

B. BG self-adjoint extension approach

As mentioned above, the KS approach is suitable to address only bound states. On the other hand, the BG method is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameter. By comparing the results of these two approaches for bound states, the self-adjoint extension parameter can be determined in terms of the physics of the problem.

In the BG approach, the boundary condition is a mathematical limit allowing divergent solutions of the Hamiltonian H_0 at isolated points, provided they remain square integrable. All the self-adjoint extensions H_{0,λ_m} of \tilde{H}_0 are parametrized by the boundary condition at the origin [50, 52]

$$\psi^{(0)} = \lambda_m \psi^{(1)}, \quad (33)$$

with

$$\begin{aligned} \psi^{(0)} &= \lim_{r \rightarrow 0^+} r^{|m-\eta|} \psi(r), \\ \psi^{(1)} &= \lim_{r \rightarrow 0^+} \frac{1}{r^{|m-\eta|}} \left[\psi(r) - \psi^{(0)} \frac{1}{r^{|m-\eta|}} \right], \end{aligned}$$

where λ_m is the self-adjoint extension parameter. In [52] is shown that there is a relation between the self-adjoint extension parameter λ_m and the number θ in the KS approach. The number θ is associated with the mapping of deficiency subspaces and extend the domain of operator to make it self-adjoint. The self-adjoint extension parameter λ_m have a physical interpretation: it represents the scattering length [60] of H_{0,λ_m} [52]. For $\lambda_m = 0$, we have

the free Hamiltonian (without the δ function) with regular wave functions at origin and for $\lambda_m \neq 0$ the boundary condition in (33) allows a $r^{-|m-\eta|}$ singularity in the wave functions at origin.

IV. BOUND STATE ANALYSIS

In this section we employ the KS approach for determination of the bound states for the Hamiltonian in H . Thus, the first term of Eq. (32) is obtained by integrating the Eq. (30) from 0 to a . The second term is calculated using the asymptotic representation for the Bessel function $K_{|m-\eta|}$ for small argument. So, from (32) we arrive at

$$a \dot{\Upsilon}_\theta(a) + s\eta \Upsilon_\theta = 0, \quad (34)$$

with

$$\Upsilon_\theta(r) = D(\varepsilon_+) + e^{i\theta} D(\varepsilon_-), \quad (35)$$

and

$$D(\varepsilon_\pm) = \frac{(\varepsilon_\pm r)^{-|m-\eta|}}{2^{-|m-\eta|} \Gamma(1 - |m-\eta|)} - \frac{(\varepsilon_\pm r)^{|m-\eta|}}{2^{|m-\eta|} \Gamma(1 + |m-\eta|)}. \quad (36)$$

Eq. (34) gives us the parameter θ in terms of the physics of the problem, i.e., the correct behavior of the wave functions at the origin.

Next, we will find the bound states of the Hamiltonian H_0 and, by using (34), the spectrum of H will be determined without any arbitrary parameter. Then, from $H_0 \psi_\theta = E_b \psi_\theta$ we achieve the modified Bessel equation ($\kappa^2 = -2ME_b$)

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\eta)^2}{r^2} - \kappa^2 \right] \psi_\theta(r) = 0, \quad (37)$$

where $E_b < 0$ (since we are looking for bound states). The general solution for the above equation is

$$\psi_\theta(r) = K_{|m-\eta|} \left(r \sqrt{-2ME_b} \right). \quad (38)$$

Since these solutions belong to $\mathcal{D}(H_{\theta,0})$, they present the form (28) for a θ selected from the physics of the problem (cf. Eq. (34)). So, we substitute (38) into (28) and compute $a \dot{\psi}_\theta / \psi_\theta|_{r=a}$. After a straightforward calculation, we have the relation

$$\frac{|m-\eta| [a^{2|m-\eta|} (-ME_b)^{|m-\eta|} \Theta - 1]}{a^{2|m-\eta|} (-ME_b)^{|m-\eta|} \Theta + 1} = -s\eta, \quad (39)$$

where $\Theta = \Gamma(-|m-\eta|) / (2^{|m-\eta|} \Gamma(|m-\eta|))$. Solving the above equation for E_b , we find the sought energy spectrum

$$E_b = -\frac{2}{Ma^2} \left[\left(\frac{s\eta - |m-\eta|}{s\eta + |m-\eta|} \right) \frac{\Gamma(1 + |m-\eta|)}{\Gamma(1 - |m-\eta|)} \right]^{1/|m-\eta|}. \quad (40)$$

In the above relation, to ensure that the energy is a real number, we must have $|s\eta| \geq |m - \eta|$, and due to $|m - \eta| < 1$ it is sufficient to consider $|s\eta| \geq 1$. A necessary condition for a δ function generating an attractive potential, able to support bound states, is that the coupling constant must be negative. Thus, the existence of bound states with real energies requires

$$s\eta \geq 1. \quad (41)$$

From the above equation and Eq (16) it follows that $s\lambda gV_z$ must be positive, and consequently, there is a minimum value for this product.

V. SCATTERING ANALYSIS

In this section, we are interested in a situation in which an incident particle reaches the center at $r = 0$ and is scattered out by the potential $-s\eta\delta(r)/r$. The phase-shift, scattering amplitude and so on, are obtained by employing the BG approach. The equation to be solved is the Eq. (22) and its solution in the $r \neq 0$ region can be written as

$$\psi_m(r) = a_m J_{|m-\eta|}(kr) + b_m Y_{|m-\eta|}(kr), \quad (42)$$

with a_m and b_m being constants and $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of first and second kind, respectively. Upon replacing $\psi_m(r)$ in the boundary condition (33), one obtain

$$\lambda_m a_m \mathcal{A} k^{|m-\eta|} = b_m \left(\mathcal{B} k^{-|m-\eta|} - \lambda_m \mathcal{C} k^{|m-\eta|} - \lambda_m \mathcal{B} \mathcal{D} k^{-|m-\eta|} \lim_{r \rightarrow 0^+} r^{2-2|m-\eta|} \right), \quad (43)$$

$$(44)$$

with

$$\begin{aligned} \mathcal{A} &= \frac{1}{2^{|m-\eta|} \Gamma(1 + |m-\eta|)}, \\ \mathcal{B} &= -\frac{2^{|m-\eta|} \Gamma(|m-\eta|)}{\pi}, \\ \mathcal{C} &= -\frac{\cos(\pi|m-\eta|) \Gamma(-|m-\eta|)}{\pi 2^{|m-\eta|}}, \\ \mathcal{D} &= \frac{k^2}{4(1 - |m-\eta|)}. \end{aligned} \quad (45)$$

In Eq. (44), $\lim_{r \rightarrow 0^+} r^{2-2|m-\eta|}$ is divergent if $|m - \eta| \geq 1$, hence b_m must be zero. On the other hand, $\lim_{r \rightarrow 0^+} r^{2-2|m-\eta|}$ is finite for $|m - \eta| < 1$. This means that there arises the contribution of the irregular solution $Y_{|m-\eta|}(kr)$. Here, the presence of an irregular solution contributing to the wave function stems from the fact the Hamiltonian H_0 is not a self-adjoint operator when $|m - \eta| < 1$ (cf., Section III). Hence such irregular solution must be associated with a self-adjoint extension of the operator H_0 [61, 62]. Thus, for $|m - \eta| < 1$, we have

$$\lambda_m a_m \mathcal{A} k^{|m-\eta|} = b_m (\mathcal{B} k^{-|m-\eta|} - \lambda_m \mathcal{C} k^{|m-\eta|}), \quad (46)$$

and by substituting the values of \mathcal{A} , \mathcal{B} and \mathcal{C} into above expression we find

$$b_m = -\mu_m^{\lambda_m}(k, \eta) a_m, \quad (47)$$

where

$$\mu_m^{\lambda_m}(k, \eta) = \frac{\lambda_m k^{2|m-\eta|} \Gamma(1 - |m-\eta|) \sin(\pi|m-\eta|)}{B_k}, \quad (48)$$

and

$$\begin{aligned} B_k &= \lambda_m k^{2|m-\eta|} \Gamma(1 - |m-\eta|) \cos(\pi|m-\eta|) \\ &+ 4^{|m-\eta|} \Gamma(1 + |m-\eta|). \end{aligned} \quad (49)$$

Since a δ function is a very short range potential, it follows that the asymptotic behavior of $\psi_m(r)$ for $r \rightarrow \infty$ is given by [63]

$$\psi_m(r) \sim \sqrt{\frac{2}{\pi kr}} \cos \left(kr - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_m^{\lambda_m}(k, \eta) \right), \quad (50)$$

where $\delta_m^{\lambda_m}(k, \eta)$ is a scattering phase shift. The phase shift is a measure of the argument difference to the asymptotic behavior of the solution $J_{|m|}(kr)$ of the radial free equation which is regular at the origin. By using the asymptotic behavior of the Bessel functions [64] into Eq. (42) one obtain

$$\begin{aligned} \psi_m(r) \sim \sqrt{\frac{2}{\pi kr}} \left[\cos \left(kr - \frac{\pi|m-\eta|}{2} - \frac{\pi}{4} \right) \right. \\ \left. - \mu_m^{\lambda_m}(k, \eta) \sin \left(kr - \frac{\pi|m-\eta|}{2} - \frac{\pi}{4} \right) \right]. \end{aligned} \quad (51)$$

By comparing the above expression with Eq. (50), we have

$$\delta_m^{\lambda_m}(k, \eta) = \Delta_m(\eta) + \theta_m^{\lambda_m}(k, \eta), \quad (52)$$

with

$$\Delta_m(\eta) = \frac{\pi}{2} (|m| - |m-\eta|), \quad (53)$$

the phase shift of the AC scattering and

$$\theta_m^{\lambda_m}(k, \eta) = \arctan [\mu_m^{\lambda_m}(k, \eta)]. \quad (54)$$

Therefore, the scattering operator $S_m^{\lambda_m}(k, \eta)$ (S -matrix) for the self-adjoint extension is

$$S_m^{\lambda_m}(k, \eta) = e^{2i\delta_m^{\lambda_m}(k, \eta)} = \left[\frac{1 + i\mu_m^{\lambda_m}(k, \eta)}{1 - i\mu_m^{\lambda_m}(k, \eta)} \right] e^{2i\Delta_m(\eta)}. \quad (55)$$

Using Eq. (48), we have

$$S_m^{\lambda_m}(k, \eta) = e^{2i\Delta_m(\eta)} \frac{\Omega_+}{\Omega_-}, \quad (56)$$

with

$$\Omega_{\pm} = B_k \pm i\lambda_m k^{2|m-\eta|} \Gamma(1 - |m-\eta|) \sin(i\pi|m-\eta|). \quad (57)$$

Hence, for any value of the self-adjoint extension parameter λ_m , there is an additional scattering. If $\lambda_m = 0$, we achieve the corresponding result for the AC problem with Dirichlet boundary condition [55], i.e., $S_m^0(k, \eta) = e^{2i\Delta_m(\eta)}$. If we make $\lambda_m = \infty$, we get $S_m^\infty(k, \eta) = e^{2i\Delta_m(\eta) + 2i\pi|m-\eta|}$.

In accordance with the general theory of scattering, the poles of the S -matrix in the upper half of the complex plane [65] determine the positions of the bound states in the energy scale. These poles occur in the denominator of (56) with the replacement $k \rightarrow i\kappa$. Thus,

$$\Omega_- = 0. \quad (58)$$

Solving the above equation for E_b , we found the bound state energy

$$E_b = -\frac{2}{M} \left[-\frac{1}{\lambda_m} \frac{\Gamma(1+|m-\eta|)}{\Gamma(1-|m-\eta|)} \right]^{1/|m-\eta|}, \quad (59)$$

for $\lambda_m < 0$. Hence, the poles of the scattering matrix only occur for negative values of the self-adjoint extension parameter, when we have scattering and bound states. In this latter case, the scattering operator can be expressed in terms of the bound state energy

$$S_m^{\lambda_m}(k, \eta) = e^{2i\Delta_m(\eta)} \left[\frac{e^{2i\pi|m-\eta|} - (\kappa/k)^{2|m-\eta|}}{1 - (\kappa/k)^{2|m-\eta|}} \right]. \quad (60)$$

By comparing Eq. (59) with the Eq. (40) we have

$$\frac{1}{\lambda_m} = -\frac{1}{a^{2|m-\eta|}} \left(\frac{s\eta - |m-\eta|}{s\eta + |m-\eta|} \right). \quad (61)$$

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the problem. It should be mentioned that some relations involving the self-adjoint extension parameter and the δ -function coupling constant were previously obtained by using Green's function in Ref. [66] and the renormalization technique in Ref. [67], being both, however, deprived from a clear physical interpretation.

The scattering amplitude $f(k, \eta)$ can be now obtained using the standard methods of scattering theory, namely ($f_k = 1/\sqrt{2\pi ik}$)

$$\begin{aligned} f(k, \eta) &= f_k \sum_{m=-\infty}^{\infty} \left(e^{2i\delta_m^{\lambda_m}(k, \eta)} - 1 \right) e^{im\varphi} \\ &= f_k \left\{ \sum_{|m-\eta| \geq 1} (e^{2i\Delta_m(\eta)} - 1) e^{im\varphi} \right. \\ &\quad + \sum_{|m-\eta| < 1} \left\{ e^{2i\Delta_m(\eta)} \left[\frac{1 + i\mu_m^{\lambda_m}(k, \eta)}{1 - i\mu_m^{\lambda_m}(k, \eta)} \right] - 1 \right\} \\ &\quad \left. \times e^{im\varphi} \right\}. \end{aligned} \quad (62)$$

The first sum is the AC amplitude (i.e., in the absence of the δ function), while the second sum is the contribution

that come from the singular solutions. In the above equation we can see that the scattering amplitude is energy dependent (cf., Eq. (48)). This is a clearly manifestation of the known non-conservation of the helicity in the AC scattering [53], because the only length scale in the nonrelativistic problem is set by $1/k$, so it follows that the scattering amplitude would be a function of the angle alone, multiplied by $1/k$ [68]. In fact, the failure of helicity conservation expressed in Eq. (62), it stems from the fact that the δ function singularity make the Hamiltonian and the helicity nonself-adjoint operators [69–72]. By expressing the helicity operator, $\hat{h} = \mathbf{\Sigma} \cdot \mathbf{\Pi}$, in terms of the variables used in (21), we attain

$$\hat{h} = \begin{pmatrix} 0 & -i \left(\partial_r + \frac{|m-\eta|+1}{r} \right) \\ -i \left(\partial_r - \frac{|m-\eta|}{r} \right) & 0 \end{pmatrix}. \quad (63)$$

Notice that under a parity π transformation $\hat{h} \rightarrow \pi^\dagger \hat{h} \pi = -\hat{h}$, that comes immediately from the parity transformation $\pi^\dagger r \pi = -r$. This is in fact the helicity odd-parity property. The helicity operator share the same issue as the Hamiltonian operator in the interval $|m-\eta| < 1$, i.e., it is not self-adjoint [73, 74]. Despite that on a finite interval $[0, L]$, \hat{h} is a self-adjoint operator with domain in the functions satisfying $\xi(L) = e^{i\theta} \xi(0)$, it does not admit a self-adjoint extension on the interval $[0, \infty)$ [75], and consequently it can be not conserved, thus the helicity conservation is broken due to the presence of the singularity at the origin [68, 70].

VI. CONCLUSION

We have studied the spin-1/2 AC bound and scattering problem with a Lorentz-violating and CPT-odd nonminimal coupling between fermions and the gauge field in the context of the Dirac equation. The self-adjoint extension approach was used to determine the bound states of the particle in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter. It has been shown that there is an additional scattering for any value of the self-adjoint extension parameter and for negative values of this parameter there is bound states. By comparing the results from bound and scattering scenarios, the self-adjoint extension parameter was determined. The scattering amplitude show a energy dependency, so the helicity is not conserved. This stem from the fact that the helicity operator is not a self-adjoint extension operator. Therefore, it does not represent a quantum observable and does not correspond to a conserved quantity.

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